MAXIMAL CLONES ON UNCOUNTABLE SETS THAT INCLUDE ALL PERMUTATIONS

MICHAEL PINSKER

Abstract. We first determine the maximal clones on a set X of infinite regular cardinality κ which contain all permutations but not all unary functions, extending a result of Heindorf's for countably infinite X. If κ is countably infinite or weakly compact, this yields a list of all maximal clones containing the permutations since in that case the maximal clones above the unary functions are known. We then generalize a result of Gavrilov's to obtain on all infinite X a list of all maximal submonoids of the monoid of unary functions which contain the permutations.

1. Clones and the Results

1.1. **The clone lattice.** Let X be a set of size $|X| = \kappa$. For each natural number $n \geq 1$ we denote the set of functions on X of arity n by $\mathscr{O}^{(n)}$. We set $\mathscr{O} = \bigcup_{n=1}^{\infty} \mathscr{O}^{(n)}$ to be the set of all finitary operations on X. A *clone* is a subset of $\mathscr{O}$ which contains the projection maps and which is closed under composition. Since arbitrary intersections of clones are obviously again clones, the set of clones on X forms a complete algebraic lattice $Cl(X)$ which is a subset of the power set of $\mathscr{O}$. The clone lattice is countably infinite if X has exactly two elements, and of size continuum if X is finite and has at least three elements. On infinite X we have $|Cl(X)| = 2^{2^\kappa}$.

The dual atoms of the clone lattice are called *maximal* clones. On finite X there exist finitely many maximal clones and an explicit list of those clones has been provided by Rosenberg [6]. Moreover, the clone lattice is dually atomic in that case, that is, every clone is contained in a maximal one. For X infinite the number of maximal clones equals the size of the whole clone lattice ([8], see also [2]), so that it seems impossible to know all of them. It has also been shown [3] that if the continuum hypothesis holds, then not every clone on a countably infinite set is contained in a maximal one.

1.2. **Clones containing the bijections.** However, even on infinite X the sublattice of $Cl(X)$ of clones containing the set $\mathscr{S}$ of all permutations of X

1991 *Mathematics Subject Classification.* Primary 08A40; secondary 08A05.

Key words and phrases. clone lattice, permutations, maximal clones, almost unary functions, unary clones, maximal monoids.

The author would like to thank L. Heindorf for his many comments on earlier versions of the paper. Support by DOC [Doctoral Scholarship Programme of the Austrian Academy of Sciences] is gratefully acknowledged.

is dually atomic since $\mathscr{O}$ is finitely generated over $\mathscr{S}$: Call a set $A \subseteq X$ *large* iff $|A| = |X| = \kappa$ and *small* otherwise. Moreover, A is *co-large* iff $X \setminus A$ is large, and *co-small* iff $X \setminus A$ is small. Set

$$\mathscr{I} = \{f \in \mathscr{O}^{(1)} : f \text{ is injective and } f[X] \text{ is co-large}\}$$

and

$$\mathscr{J} = \{g \in \mathscr{O}^{(1)} : g^{-1}[y] \text{ is large for all } y \in X\}.$$

It is readily verified that for arbitrary fixed $f \in \mathscr{I}$ and $g \in \mathscr{J}$ we have

$$\mathscr{I} = \{\alpha \circ f : \alpha \in \mathscr{S}\} \text{ and } \mathscr{J} = \{\alpha \circ g \circ \beta : \alpha, \beta \in \mathscr{S}\}.$$

Moreover,

$$\mathscr{O}^{(1)} = \{j \circ i : j \in \mathscr{J}, i \in \mathscr{I}\}.$$

Together with the well-known fact that $\mathscr{O}^{(1)} \cup \{p\}$ generates $\mathscr{O}$ for any binary injection p we conclude that $\mathscr{O}$ is generated by $\mathscr{S} \cup \{p, f, g\}$. Hence Zorn's lemma implies that the interval $[\mathscr{S}, \mathscr{O}]$ is dually atomic.

We will determine all maximal clones $\mathscr{C}$ on a base set of regular cardinality for which $\mathscr{S} \subseteq \mathscr{C}$ but not $\mathscr{O}^{(1)} \subseteq \mathscr{C}$. This has already been done for countable base sets by Heindorf in the article [4] in the following way: Let $\rho \subseteq X^J$ be a relation on X indexed by J and let $f \in \mathscr{O}^{(n)}$. We say that f *preserves* ρ iff for all $r^1 = (r^1_i : i \in J), \ldots, r^n = (r^n_i : i \in J)$ in ρ we have $(f(r^1_i, \ldots, r^n_i) : i \in J) \in \rho$. We define the clone of *polymorphisms* $\mathrm{Pol}(\rho)$ of $\rho \subseteq X^J$ to consist exactly of the functions in $\mathscr{O}$ preserving ρ. In particular, if $\rho \subseteq X^{X^k}$ is a set of k-ary functions, then the polymorphisms of ρ are exactly those $f \in \mathscr{O}^{(n)}$ for which the composite $f(g_1, \ldots, g_n) \in \rho$ whenever $g_1, \ldots, g_n \in \rho$. Now it is obvious that since clones are closed under composition we have $\mathscr{C} \subseteq \mathrm{Pol}(\mathscr{C}^{(n)})$ for any clone $\mathscr{C}$ and for all $n \geq 1$, where $\mathscr{C}^{(n)} = \mathscr{C} \cap \mathscr{O}^{(n)}$. Moreover, $\mathrm{Pol}(\mathscr{C}^{(n)})^{(n)} = \mathscr{C}^{(n)}$. Therefore, if $\mathscr{C}$ is a maximal clone such that $\mathscr{S} \subseteq \mathscr{C}^{(1)} \subsetneqq \mathscr{O}^{(1)}$, then $\mathscr{C} \subseteq \mathrm{Pol}(\mathscr{C}^{(1)}) \subsetneqq \mathscr{O}$ holds. Hence $\mathscr{C} = \mathrm{Pol}(\mathscr{C}^{(1)})$ by the maximality of $\mathscr{C}$. We conclude that all maximal clones with $\mathscr{S} \subseteq \mathscr{C}^{(1)} \subsetneqq \mathscr{O}^{(1)}$ are of the form $\mathrm{Pol}(\mathscr{G})$, where $\mathscr{S} \subseteq \mathscr{G} \subsetneqq \mathscr{O}^{(1)}$ is a *submonoid* of $\mathscr{O}^{(1)}$, that is, a set of unary functions closed under composition and containing the identity map.

We say that a property holds for *almost all* $y \in X$ iff the set of all elements for which the property does not hold is small. For $\lambda \leq \kappa$ a cardinal define a unary function f to be λ-*surjective* iff $|X \setminus f[X]| < \lambda$. Instead of κ-surjective we also say *almost surjective*; this means that the range of f is co-small. f is λ-*injective* iff $|\{x \in X : \exists y \neq x\, (f(x) = f(y))\}| < \lambda$. For $\lambda = 1$ or infinite, this is the case iff there exists a set $A \subseteq X$ such that $|A| < \lambda$ and such that the restriction of f to the complement of A is injective. *Almost injective* means κ-injective.

We are going to prove the following theorem.

Theorem 1. *Let X be a set of regular cardinality κ. The maximal clones over X which contain all bijections but not all unary functions are exactly*

those of the form $\mathrm{Pol}(\mathscr{G})$*, where* $\mathscr{G} \in \{\mathscr{A}, \mathscr{B}, \mathscr{E}, \mathscr{F}\} \cup \{\mathscr{G}_\lambda : 1 \leq \lambda \leq \kappa, \lambda \text{ a cardinal}\}$ *is one of the following submonoids of* $\mathscr{O}^{(1)}$*:*

1. $\mathscr{A} = \{f \in \mathscr{O}^{(1)} : f^{-1}[y] \text{ is small for almost all } y \in X\}$
2. $\mathscr{B} = \{f \in \mathscr{O}^{(1)} : f^{-1}[y] \text{ is small for all } y \in X\}$
3. $\mathscr{E} = \{f \in \mathscr{O}^{(1)} : f \text{ is almost surjective}\}$
4. $\mathscr{F} = \{f \in \mathscr{O}^{(1)} : f \text{ is almost surjective or constant}\}$
5. $\mathscr{G}_\lambda = \{f \in \mathscr{O}^{(1)} : \text{if } A \subseteq X \text{ has cardinality } \lambda \text{ then } |X \setminus f[X \setminus A]| \geq \lambda\}$

Corollary 2. *Let* X *be a set of regular cardinality* $\kappa = \aleph_\alpha$*. Then there exist* $\max(|\alpha|, \aleph_0)$ *maximal clones on* X *which contain all bijections but not all unary functions.*

On some infinite sets, namely countably infinite ones and sets of weakly compact cardinality, it is known that there exist exactly two maximal clones $\mathrm{Pol}(T_1)$ and $\mathrm{Pol}(T_2)$ which contain $\mathscr{O}^{(1)}$ (T_1 and T_2 are certain sets of binary functions). See Gavrilov [1] for the countable, and Goldstern and Shelah [2] for the uncountable. Hence in those cases, our theorem completes the list of maximal clones above $\mathscr{S}$. It is a fact that weakly compact cardinals κ satisfy $\kappa = \aleph_\kappa$. Thus we have

Corollary 3. *Let* X *be a set of countably infinite or weakly compact cardinality* κ*. Then there exist* κ *maximal clones which contain all bijections.*

Unfortunately things are not always that easy, as demonstrated by Goldstern and Shelah in [2]: For many regular cardinalities of X, in particular for all successors of uncountable regulars, there exist 2^{2^κ} maximal clones which contain $\mathscr{O}^{(1)}$. It is interesting that whereas above $\mathscr{O}^{(1)}$ the number of maximal clones varies heavily with the partition properties of the underlying base set (2 for weakly compact cardinals, 2^{2^κ} for many others), the number of maximal clones above the permutations but not above $\mathscr{O}^{(1)}$ is a monotone function of κ and always relatively small ($\leq \kappa$).

1.3. **Maximal submonoids of $\mathscr{O}^{(1)}$.** Not all monoids appearing in Theorem 1 are maximal submonoids of $\mathscr{O}^{(1)}$ (by a maximal submonoid of $\mathscr{O}^{(1)}$ we mean a dual atom in the lattice of submonoids of $\mathscr{O}^{(1)}$ with inclusion). More surprisingly, there exist maximal submonoids of $\mathscr{O}^{(1)}$ above the permutations whose polymorphism clone is not maximal. Observe that submonoids of $\mathscr{O}^{(1)}$ differ only formally from *unary clones*, that is clones consisting only of essentially unary functions, and that the lattice of monoids which contain the permutations is dually atomic by the argument we have seen before. We are going to prove the following theorem for arbitrary infinite sets in the last section, generalizing a theorem due to Gavrilov [1] for countable base sets.

Theorem 4. *Let* X *be an infinite set. If* X *has regular cardinality, then the maximal submonoids of* $\mathscr{O}^{(1)}$ *which contain the permutations are exactly the monoid* $\mathscr{A}$ *and the monoids* $\mathscr{G}_\lambda$ *and* $\mathscr{M}_\lambda$ *for* $\lambda = 1$ *and* $\aleph_0 \leq \lambda \leq \kappa$*,* λ *a*

cardinal, where

$$\mathscr{M}_\lambda = \{f \in \mathscr{O}^{(1)} : f \text{ is } \lambda\text{-surjective or not } \lambda\text{-injective}\}.$$

If X has singular cardinality, then the same is true with the monoid $\mathscr{A}$ replaced by

$$\mathscr{A}' = \{f \in \mathscr{O}^{(1)} : \exists \lambda < \kappa \ (\,|f^{-1}[\{x\}]| \leq \lambda \text{ for almost all } x \in X\,)\,\}.$$

Corollary 5. *On a set X of infinite cardinality $\aleph_\alpha$ there exist $2\,|\alpha|+5$ maximal submonoids of $\mathscr{O}^{(1)}$ that contain the permutations. Hence the smallest cardinality on which there are infinitely many such monoids is $\aleph_\omega$.*

Observe that the statement about singular cardinals in Theorem 4 differs only slightly from the corresponding one for regulars. We do not know whether Theorem 1 can be generalized to singulars, but in our proof we do use the regularity condition (in Proposition 8, Lemma 9 and permanently in Section 2.2).

1.4. **An equivalent definition of $\mathscr{G}_\lambda$.** In the countable versions of Theorems 1 and 4 a different but equivalent definition of $\mathscr{G}_{\aleph_0}$ was used: Define for $\lambda = 1$ and for all $\aleph_0 \leq \lambda \leq \kappa$ monoids

$$\delta(\lambda) = \{f \in \mathscr{O}^{(1)} : f \text{ is } \lambda\text{-injective or not } \lambda\text{-surjective}\}$$

(this definition and notation is due to Rosenberg [7]). Then we have

Lemma 6. $\delta(\lambda) = \mathscr{G}_\lambda$ *for* $\lambda = 1$ *and* $\aleph_0 \leq \lambda \leq \kappa$.

Proof. Note that for $\lambda = 1$, λ-injective simply means injective and λ-surjective means surjective. The lemma is easily verified for that case, and we prove it for λ infinite.

Assuming $f \in \delta(\lambda)$ we show $f \in \mathscr{G}_\lambda$. It is clear that if f is not λ-surjective, then $f \in \mathscr{G}_\lambda$. So assume f is λ-surjective; then by the definition of $\delta(\lambda)$, f is λ-injective. Now let $A \subseteq X$ be an arbitrary set of size λ. Assume towards contradiction that $|X \setminus f[X \setminus A]| < \lambda$. Then two things can happen: If $|f[A] \cap f[X \setminus A]| \geq \lambda$, then $|\{x \in X : \exists y \neq x\,(f(x) = f(y))\}| \geq |\{x \in A : \exists y \in X \setminus A\,(f(x) = f(y))\}| \geq \lambda$, contradicting the λ-injectivity of f. Otherwise, A is mapped onto a set of size smaller than λ, again in contradiction to f being λ-injective.

To see the other inclusion, take any $f \notin \delta(\lambda)$. Then f is not λ-injective; thus we can find $A \subseteq X$ of size λ such that $f[X] = f[X \setminus A]$. But then $|X \setminus f[X \setminus A]| = |X \setminus f[X]| < \lambda$ as f is λ-surjective. Hence, $f \notin \mathscr{G}_\lambda$. □

Before we start with the proofs we fix some global notation.

1.5. **Notation.** For a set of functions $\mathscr{F}$ we shall denote the smallest clone containing $\mathscr{F}$ by $\langle \mathscr{F} \rangle$. We call the projections which every clone contains π^n_i, where $n \geq 1$ and $1 \leq i \leq n$. We write n_f for the arity of a function $f \in \mathscr{O}$ whenever that arity has not yet been given another name. If $a \in X^n$ is an n-tuple and $1 \leq k \leq n$ we write a_k for the k-th component of a.

The image of a set $A \subseteq X^n$ under a function $f \in \mathscr{O}^{(n)}$ we denote by $f[A]$. Similarly we write $f^{-1}[A]$ for the preimage of $A \subseteq X$ under f. If $A = \{c\}$ is a singleton we cut short and write $f^{-1}[c]$ rather than $f^{-1}[\{c\}]$. Occasionally we shall denote the constant function with value $c \in X$ also by c. Whenever we identify X with its cardinality we let $<$ and $\leq$ refer to the canonical well-order on X.

2. The proof of Theorem 1

In this section we are going to prove Theorem 1; it will be the direct consequence of Propositions 8, 10, 11, 16, 20, 21, 22, and 28. The first part of the proof (Section 2.1) is not much more than a translation of Heindorf's paper [4] to arbitrary regular cardinals; the reader familiar with that article should not be surprised to find the same constructions here. In Section 2.2 we have to go an own way to finish the proof.

2.1. **The beginning of the proof.** We start with a general observation which will be useful.

Lemma 7. *Let $\mathscr{G}$ be a proper submonoid of $\mathscr{O}^{(1)}$ such that $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$ for all unary $h \notin \mathscr{G}$. Then $\mathrm{Pol}(\mathscr{G})$ is maximal.*

Proof. Let $f \notin \mathrm{Pol}(\mathscr{G})$ be given. Then there exist $h_1, \ldots, h_{n_f} \in \mathscr{G}$ such that $h = f(h_1, \ldots, h_{n_f}) \notin \mathscr{G}$. Now $h \in \langle \mathscr{G} \cup \{f\} \rangle \subseteq \langle \mathrm{Pol}(\mathscr{G}) \cup \{f\} \rangle$ and $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$ by assumption so that we conclude $\langle \mathrm{Pol}(\mathscr{G}) \cup \{f\} \rangle = \mathscr{O}$. ◻

2.1.1. *The monoids $\mathscr{A}$ and $\mathscr{B}$.*

Proposition 8. *The clones $\mathrm{Pol}(\mathscr{A})$ and $\mathrm{Pol}(\mathscr{B})$ are maximal.*

Proof. The maximality of $\mathrm{Pol}(\mathscr{A})$ has been proved in [1] for the countable case and in [7] (Proposition 4.1) for arbitrary sets of regular cardinality (although not stated there, the regularity condition is necessary, for otherwise $\mathscr{A}$ is not closed under composition).

For the maximality of $\mathrm{Pol}(\mathscr{B})$, let a unary $h \notin \mathscr{B}$ be given; by Lemma 7, it suffices to show $\langle \mathrm{Pol}(\mathscr{G}) \cup \{h\} \rangle = \mathscr{O}$. By the definition of $\mathscr{B}$ there exists $c \in X$ such that the preimage $Y = h^{-1}[c]$ is large. Choose any injection $g : X \to Y$; then $h \circ g(x) = c$ for all $x \in X$.

Now let $f \in \mathscr{O}^{(n)}$ be an arbitrary function and consider $\tilde{f} \in \mathscr{O}^{(n+1)}$ defined by

$$\tilde{f}(x_1, \ldots, x_n, y) = \begin{cases} f(x_1, \ldots, x_n) & , y = c \\ y & , y \neq c. \end{cases}$$

We claim that $\tilde{f} \in \mathrm{Pol}(\mathscr{B})$. For let $\alpha_1, \ldots, \alpha_n, \beta \in \mathscr{B}$ and $d \in X$ be given. If $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)(x) = d$, then by the definition of $\tilde{f}$ either $\beta(x) = c$ and $f(\alpha_1(x), \ldots, \alpha_n(x)) = d$ or $\beta(x) \neq c$ and $\beta(x) = d$. But since $\beta \in \mathscr{B}$, the set of all $x \in X$ such that $\beta(x) = c$ or $\beta(x) = d$ is small. Hence $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)^{-1}[d]$ is small and so $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta) \in \mathscr{B}$.

Now to finish the proof it is enough to observe that $f(x_1,\dots,x_n) = \tilde{f}(x_1,\dots,x_n,c) = \tilde{f}(x_1,\dots,x_n,h\circ g(x_1)) \in \langle \mathrm{Pol}(\mathscr{B}) \cup \{h\}\rangle$. $\square$

We will prove now that $\mathscr{B}$ is the only proper submonoid of $\mathscr{A}$ whose Pol is maximal.

Lemma 9. *If $f \notin \mathrm{Pol}(\mathscr{A})$, then there exist $\alpha_1,\dots,\alpha_{n_f} \in \mathscr{O}^{(1)}$ constant or injective such that $f(\alpha_1,\dots,\alpha_{n_f}) \notin \mathscr{A}$.*

Proof. Since $f \notin \mathrm{Pol}(\mathscr{A})$, there exist $\beta_1,\dots,\beta_{n_f} \in \mathscr{A}$ such that $f(\beta_1,\dots,\beta_{n_f}) \notin \mathscr{A}$. We will use induction over n_f. If $n_f = 1$, then $f \notin \mathrm{Pol}(\mathscr{A})^{(1)} = \mathscr{A}$ so that $f(\pi^1_1) = f \notin \mathscr{A}$ which proves the assertion for that case. Now assume the lemma holds for all functions of arity at most $n_f - 1$. Define for $1 \le i \le n_f$ sets $B_i = \{y \in X : \beta_i^{-1}[y] \text{ is large}\}$. By definition of $\mathscr{A}$, all B_i are small. Set

$$\Gamma = (\beta_1,\dots,\beta_{n_f})[X] \setminus \prod_{1\le i\le n_f} B_i \subseteq X^{n_f}$$

Claim. There exists a large set $D \subseteq X$ such that $f^{-1}[d] \cap \Gamma$ is large for all $d \in D$.

To prove the claim, set $D = \{d \in X : f(\beta_1,\dots,\beta_{n_f})^{-1}[d] \text{ large}\} \setminus f[\prod_{1\le i\le n_f} B_i]$. The set D is large as $f(\beta_1,\dots,\beta_{n_f}) \notin \mathscr{A}$ and as $\prod_{1\le i\le n_f} B_i$ is small. Define $A_d = (f(\beta_1,\dots,\beta_{n_f}))^{-1}[d]$ for each $d \in D$. Then $(\beta_1,\dots,\beta_{n_f})[A_d] \subseteq \Gamma$ is large for all $d \in D$. Indeed, assume to the contrary that there exists $d \in D$ such that $(\beta_1,\dots,\beta_{n_f})[A_d]$ is small; then, since $|X| = \kappa$ is regular, there is an $x \in (\beta_1,\dots,\beta_{n_f})[A_d]$ so that $(\beta_1,\dots,\beta_{n_f})^{-1}[x]$ is large. But then we would have $x \in \prod_{1\le i\le n_f} B_i$, in contradiction to the assumption that $d \notin f[\prod_{1\le i\le n_f} B_i]$. This proves the claim since $f^{-1}[d] \cap \Gamma = (\beta_1,\dots,\beta_{n_f})[A_d]$ is large for every $d \in D$.

Defining hyperplanes $H^i_b = \{x \in X^{n_f} : x_i = b\}$ for all $1 \le i \le n_f$ and all $b \in X$, we can write Γ as follows:

$$\Gamma = (\bigcup_{i=1}^{n_f} \bigcup_{b\in B_i} \Gamma \cap H^i_b) \cup \Delta,$$

where $\Delta = \Gamma \setminus \bigcup_{i=1}^{n_f} \bigcup_{b\in B_i} H^i_b$. Since κ is regular and the union consists only of a small number of sets, we have that either there exist $1 \le i \le n_f$ and some $b \in B_i$ such that $f^{-1}[d] \cap \Gamma \cap H^i_b$ is large for a large set of $d \in D$, or $f^{-1}[d] \cap \Delta$ is large for a large set of $d \in D$. We distinguish the two cases:

Case 1. There exist $1 \le i \le n_f$ and $b \in B_i$ such that $f^{-1}[d] \cap \Gamma \cap H^i_b$ is large for many $d \in D$; say without loss of generality $i = n_f$. Then $f(\beta_1,\dots,\beta_{n_f-1},b) \notin \mathscr{A}$. By induction hypothesis, there exist $\alpha_1,\dots,\alpha_{n_f-1}$ injective or constant such that $f(\alpha_1,\dots,\alpha_{n_f-1},b) \notin \mathscr{A}$. Setting $\alpha_{n_f}(x) = b$ for all $x \in X$ proves the lemma.

Case 2. $f^{-1}[d] \cap \Delta$ is large for many $d \in D$. Observe that for all $a \in X$

and all $1 \leq i \leq n_f$, $\Delta \cap H^i_a$ is small, for otherwise $\beta_i^{-1}[a]$ would be large and thus $a \in B_i$, contradiction. Set

$$C = \{c \in X : f^{-1}[c] \cap \Delta \text{ large}\}.$$

By the assumption for this case, C is large. Now fix any $g : X \to C$ such that $g^{-1}[c]$ is large for all $c \in C$. We define a function $\alpha : X \to \Delta$ such that $f \circ \alpha = g$; moreover, $\alpha_i = \pi_i^{n_f} \circ \alpha$ will be injective, $1 \leq i \leq n_f$. Identify X with its cardinality κ. Then all α_i are injective iff $\alpha_i(x) \neq \alpha_i(y)$ for all $y < x$ and all $1 \leq i \leq n_f$. This is the case iff

$$(\alpha_1, \ldots, \alpha_{n_f})(x) \in \Delta \setminus \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}.$$

Using transfinite induction on κ, we define $(\alpha_1, \ldots, \alpha_{n_f})$ by picking

$$(\alpha_1, \ldots, \alpha_{n_f})(x) \in (f^{-1}[g(x)] \cap \Delta) \setminus \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}.$$

This is possible as $f^{-1}[g(x)] \cap \Delta$ is large for all $x \in X$ whereas $\Delta \cap \bigcup_{y<x} \bigcup_{i=1}^{n_f} H^i_{\alpha_i(y)}$ is small. Clearly $f(\alpha_1, \ldots, \alpha_{n_f}) = g \notin \mathscr{A}$ and the proof of the lemma is complete. $\square$

Proposition 10. *Let $\mathscr{G} \subseteq \mathscr{A}$ be a submonoid of $\mathscr{O}^{(1)}$ which contains all permutations. Then either $\mathscr{G} \subseteq \mathscr{B}$ or* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{A})$.

Proof. Assume $\mathscr{G} \nsubseteq \mathscr{B}$; we show $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{A})$. Observe first that for all co-large $A \subseteq X$ and all $a \in X$ there exists $g \in \mathscr{G}$ such that $g[A] = \{a\}$. Indeed, choose any $h \in \mathscr{G} \setminus \mathscr{B}$. There exists $y \in X$ such that $h^{-1}[y]$ is large. Choose bijections $\alpha, \beta \in \mathscr{S}$ with the property that $\alpha[A] \subseteq h^{-1}[y]$ and that $\beta(y) = a$. Then $g = \beta \circ h \circ \alpha$ has the desired property.
Now let $f \notin \mathrm{Pol}(\mathscr{A})$ be arbitrary; we show $f \notin \mathrm{Pol}(\mathscr{G})$. By the preceding lemma there exist $\alpha_1, \ldots, \alpha_{n_f}$ constant or injective such that $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{A}$. Choose a large and co-large $A \subseteq X$ such that $f(\alpha_1, \ldots, \alpha_{n_f})^{-1}[x] \cap A$ is large for a large set of $x \in X$. We modify the α_i to $\gamma_i \in \mathscr{G}$ in such a way that $\alpha_i \restriction_A = \gamma_i \restriction_A$ for $1 \leq i \leq n_f$: If α_i is injective, then we can choose γ_i to be a bijection. If α_i is constant, then γ_i is delivered by the observation we just made. Thus, as $f(\alpha_1, \ldots, \alpha_{n_f}) \restriction_A = f(\gamma_1, \ldots, \gamma_{n_f}) \restriction_A$ we have $f(\gamma_1, \ldots, \gamma_{n_f}) \notin \mathscr{A} \supseteq \mathscr{G}$. $\square$

Proposition 11. *Let $\mathscr{G} \subseteq \mathscr{B}$ be a submonoid of $\mathscr{O}^{(1)}$ which contains all permutations. Then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{B})$.

Proof. For arbitrary $f \notin \mathrm{Pol}(\mathscr{B})$ we show $f \notin \mathrm{Pol}(\mathscr{G})$. There are $\beta_1, \ldots, \beta_{n_f} \in \mathscr{B}$ such that there exists $c \in X$ with the property that $f(\beta_1, \ldots, \beta_{n_f})^{-1}[c]$ is large. Define $\Gamma = (\beta_1, \ldots, \beta_{n_f})[X]$. Then since $\beta_i \in \mathscr{B}$, $H^i_a \cap \Gamma$ is small for all $1 \leq i \leq n_f$ and all $a \in X$, where $H^i_a = \{x \in X^{n_f} : x_i = a\}$. Moreover, $f^{-1}[c] \cap \Gamma$ is large. Just like at the end of the proof of Lemma 9,

we can construct injective $\alpha_1, \ldots, \alpha_{n_f}$ such that $f(\alpha_1, \ldots, \alpha_{n_f})$ is constant with value c. Choose $A \subseteq X$ large and co-large and bijections $\gamma_1, \ldots, \gamma_{n_f}$ such that $\gamma_i \restriction_A = \alpha_i \restriction_A$ for $1 \leq i \leq n_f$. Then, being constant on A, $f(\gamma_1, \ldots, \gamma_{n_f}) \notin \mathscr{B} \supseteq \mathscr{G}$. Thus, $f \notin \mathrm{Pol}(\mathscr{G})$. □

2.1.2. *Generous functions.* We now turn to monoids $\mathscr{G} \supseteq \mathscr{S}$ which are not submonoids of $\mathscr{A}$. Our first goal is Proposition 16, in which we give a positive description of such monoids.

Definition 12. A function $f \in \mathscr{O}^{(1)}$ is called *generous* iff $f^{-1}[y]$ is either large or empty for all $y \in X$.

Notation 13. Let $0 \leq \lambda \leq \kappa$ be a cardinal. We denote by $\mathscr{I}_\lambda$ the set of all generous functions f with the property that $|X \setminus f[X]| = \lambda$.

The verification of the following simple facts is left to the reader.

Lemma 14. (1) *If $g \in \mathscr{O}^{(1)}$ is generous, then $f \circ g$ is generous for all $f \in \mathscr{O}^{(1)}$.*
(2) *$\mathscr{I}_\lambda$ is a subsemigroup and $\mathscr{I}_\lambda \cup \mathscr{S}$ a submonoid of $\mathscr{O}^{(1)}$ for all $\lambda \leq \kappa$.*
(3) *If $\lambda < \kappa$ and $f, g \in \mathscr{I}_\lambda$, then there exist $\alpha, \beta \in \mathscr{S}$ such that $f = \alpha \circ g \circ \beta$.*
(4) *$\mathscr{I}_\kappa$ contains all generous functions with small range, in particular the constant functions.*
(5) *If $g \in \mathscr{I}_\kappa$ has large range, then $\langle \mathscr{S} \cup \{g\} \rangle \supseteq \mathscr{I}_\kappa$.*

Lemma 15. *If $g \notin \mathscr{A}$, then there exists $\alpha \in \mathscr{S}$ such that the function $g \circ \alpha \circ g$ is generous and has large range.*

Proof. There exists a large set $A \subseteq X$ such that $g^{-1}[a]$ is large for all $a \in A$. Fix any $a_0 \in A$ and set $B_0 = g^{-1}[a_0]$. Write A as a disjoint union: $A = \{a_0\} \cup A_1 \cup A_2$, with A_1, A_2 large, and set $B_i = g^{-1}[A_i]$, $i = 1, 2$. Choose any injective partial mapping $\widetilde{\alpha}$ defined on $X \setminus A_2$ which satisfies $\widetilde{\alpha}[A_1] = B_1$ and $\widetilde{\alpha}[X \setminus (A_1 \cup A_2)] \subseteq B_0$. Since both domain and range of $\widetilde{\alpha}$ are large and co-large, we can extend the function to a bijection $\alpha \in \mathscr{S}$. Now $g \circ \alpha \circ g[X] \supseteq g \circ \alpha[A_1] = g[B_1] = A_1$, so $g \circ \alpha \circ g$ has large range. For all $x \in B_1 \cup B_2$, the equivalence class of x in the kernel of g is large, and hence also the class of x in the kernel of $g \circ \alpha \circ g$. If on the other hand $x \notin B_1 \cup B_2$, then $g(x) \notin A_1 \cup A_2$, so that $\alpha \circ g(x) \in B_0$ and therefore $g \circ \alpha \circ g(x) = a_0$. But for all $y \in B_0$ we have $g \circ \alpha \circ g(y) = a_0$ so that the kernel class of x is again large. Whence, $g \circ \alpha \circ g$ is generous. □

Proposition 16. *Let $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ be a monoid containing all bijections. Then either $\mathscr{G} \subseteq \mathscr{A}$ or there exists a cardinal $\lambda \leq \kappa$ such that $\mathscr{I}_\lambda \subseteq \mathscr{G}$.*

Proof. This is an immediate consequence of Lemmas 14 and 15. □

The preceding proposition implies that when considering submonoids $\mathscr{G}$ of $\mathscr{O}^{(1)}$ which contain the permutations, we can from now on assume that $\mathscr{I}_\lambda \subseteq \mathscr{G}$ for some λ, since we already treated the case $\mathscr{G} \subseteq \mathscr{A}$. We distinguish

two cases corresponding to the minimal λ with the property that $\mathscr{I}_\lambda \subseteq \mathscr{G}$, $\lambda > 0$ and $\lambda = 0$.

2.1.3. *The case* $0 < \lambda \leq \kappa$. We shall now investigate the case where $\mathscr{G} \not\supseteq \mathscr{I}_0$ but $\mathscr{G}$ contains $\mathscr{I}_\lambda$ for some $0 < \lambda < \kappa$. The following facts about the $\mathscr{G}_\lambda$ are left to the reader. The proof of (4) can be found in [7] (Lemma 5.2).

Lemma 17. *The following statements hold for all* $1 \leq \lambda \leq \kappa$.

1. *If* $g \in \mathscr{O}^{(n)}$ *and* $|X \setminus g[X^n]| \geq \lambda$, *then* $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$.
2. $\mathscr{G}_\lambda$ *is a submonoid of* $\mathscr{O}^{(1)}$.
3. $\mathscr{G}_n \supsetneq \mathscr{G}_{n+1}$ *for all* $1 \leq n < \aleph_0$.
4. *For* $\lambda = 1$ *and for* $\lambda \geq \aleph_0$, $\mathscr{G}_\lambda$ *is a maximal submonoid of* $\mathscr{O}^{(1)}$.

Lemma 18. *Let* $1 \leq \lambda \leq \kappa$. *If* $h \notin \mathscr{G}_\lambda$, *then there exists a* $\lambda_0 < \lambda$ *such that* $\langle \mathscr{I}_\lambda \cup \mathscr{S} \cup \{h\} \rangle \supseteq \mathscr{I}_{\lambda_0}$. *In particular,* $\langle \mathscr{G}_\lambda \cup \{h\} \rangle \supseteq \mathscr{I}_{\lambda_0}$.

Proof. There exists $A \subseteq X$, $|A| = \lambda$ such that $|X \setminus h[X \setminus A]| < \lambda$. Set $\lambda_0 = |X \setminus h[X \setminus A]|$. Choose a generous function g with $g[X] = X \setminus A$. Then $g \in \mathscr{I}_\lambda$ since $|X \setminus g[X]| = |A| = \lambda$; thus, $h \circ g \in \langle \mathscr{I}_\lambda \cup \{h\} \rangle$. On the other hand, $h \circ g \in \mathscr{I}_{\lambda_0}$ and hence $\langle \mathscr{I}_\lambda \cup \mathscr{S} \cup \{h\} \rangle \supseteq \mathscr{I}_{\lambda_0}$ by Lemma 14 (3). The second statement is a direct consequence of the inclusion $\mathscr{G}_\lambda \supseteq \mathscr{I}_\lambda \cup \mathscr{S}$. $\square$

Lemma 19. *Let* $B \subseteq X$, $|B| = \lambda_0 < \lambda \leq \kappa$, *and let* $g \in \mathscr{O}^{(2)}$ *such that* g *maps* $(X \setminus B)^2$ *bijectively onto* X *and such that* $|g[B \times X] \cup g[X \times B]| < \kappa$. *Then* $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$.

Proof. Let $\alpha, \beta \in \mathscr{G}_\lambda$ be given, and take an arbitrary $A \subseteq X$ of size λ. We have to show $|X \setminus g(\alpha, \beta)[X \setminus A]| \geq \lambda$. For $C = X \setminus \alpha[X \setminus A]$ we have $|C| \geq \lambda$. Thus, there exists some $c \in C \setminus B$. Obviously, $g(\alpha, \beta)[X \setminus A] \subseteq g[(X \setminus \{c\}) \times X]$. But the conditions on g yield that $g[(X \setminus \{c\}) \times X]$ and $g[\{c\} \times (X \setminus B)] \setminus (g[X \times B] \cup g[B \times X])$ are disjoint. Since $|g[\{c\} \times (X \setminus B)]| = \kappa$ and $|g[X \times B] \cup g[B \times X]| < \kappa$, this implies that $g(\alpha, \beta)$ misses κ values on $X \setminus A$ and hence, $g(\alpha, \beta) \in \mathscr{G}_\lambda$ and $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$. $\square$

Proposition 20. (1) $\mathrm{Pol}(\mathscr{G}_\lambda)$ *is a maximal clone for all* $1 \leq \lambda \leq \kappa$.

(2) *Let* $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ *be a monoid containing all bijections as well as some* $\mathscr{I}_\lambda$, *where* $0 \leq \lambda \leq \kappa$, *and let* λ *be minimal with this property. If* $\lambda > 0$, *then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{G}_\lambda)$.

Proof. (1) We show $\langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\} \rangle = \mathscr{O}$ for an arbitrary $h \in \mathscr{O}^{(1)} \setminus \mathscr{G}_\lambda$. By Lemma 18, there exists $\lambda_0 < \lambda$ such that $\mathscr{I}_{\lambda_0} \subseteq \langle \mathscr{G}_\lambda \cup \{h\} \rangle$. Now choose B and $g \in \mathrm{Pol}(\mathscr{G}_\lambda)$ as in Lemma 19. Consider $\alpha : X \to (X \setminus B)^2$ such that α takes every value twice. Clearly, $\alpha_1 = \pi_1^2 \circ \alpha$ and $\alpha_2 = \pi_2^2 \circ \alpha$ are elements of $\mathscr{I}_{\lambda_0}$. The function $p = g(\alpha_1, \alpha_2) = g \circ \alpha$ maps X onto X and takes every value twice as well. Therefore we can find a co-large set A such that $p[A] = X$. Now fix a mapping $q : X \to A$ so that $p \circ q$ is the identity map on X. Let an arbitrary $f \in \mathscr{O}$ be given. Then $q \circ f[X^{n_f}] \subseteq A$ is co-large which immediately implies $q \circ f \in \mathrm{Pol}(\mathscr{G}_\lambda)$. But then $f = p \circ (q \circ f) = f \in \langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\} \rangle$ and so $\langle \mathrm{Pol}(\mathscr{G}_\lambda) \cup \{h\} \rangle = \mathscr{O}$ as f was arbitrary.

(2) First we claim that $\mathscr{G} \subseteq \mathscr{G}_\lambda$. Indeed, assume there exists $h \in \mathscr{G} \setminus \mathscr{G}_\lambda$. Then, as $\mathscr{I}_\lambda \cup \mathscr{S} \subseteq \mathscr{G}$, by Lemma 18 there exists $\lambda_0 < \lambda$ such that $\mathscr{I}_{\lambda_0} \subseteq \mathscr{G}$, in contradiction to the minimality of λ. Now let $f \notin \mathrm{Pol}(\mathscr{G}_\lambda)$ be arbitrary; we prove $f \notin \mathrm{Pol}(\mathscr{G})$. There exist $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{G}_\lambda$ such that $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{G}_\lambda$. That is, there exists $A \subseteq X$ of size λ with the property that $|X \setminus f[\Gamma]| < \lambda$, where $\Gamma = \{(\alpha_1(x), \ldots, \alpha_{n_f}(x)) : x \in X \setminus A\}$. Since $\alpha_i \in \mathscr{G}_\lambda$, $1 \leq i \leq n_f$, for each i there exists a set $B_i \subseteq X$, $|B_i| = \lambda$, such that $\alpha_i[X \setminus A] \cap B_i = \emptyset$. Then $\Gamma \subseteq \Delta = (X \setminus B_1) \times \ldots \times (X \setminus B_{n_f})$. Choose $\beta : X \to \Delta$ onto and generous. Clearly $\beta_i = \pi_i^{n_f} \circ \beta \in \mathscr{I}_\lambda \subseteq \mathscr{G}$ for all $1 \leq i \leq n_f$. Now we choose any $C \subseteq X$ of size λ such that $\beta[X \setminus C] = \beta[X]$. This is possible since β is generous. Then we have that $f(\beta_1, \ldots, \beta_{n_f})[X \setminus C] = f[\Delta] \supseteq f[\Gamma]$ and so, as $|X \setminus f[\Delta]| \leq |X \setminus f[\Gamma]| < \lambda$, $f(\beta_1, \ldots, \beta_{n_f}) \notin \mathscr{G}_\lambda \supseteq \mathscr{G}$. Hence, $f \notin \mathrm{Pol}(\mathscr{G})$. □

2.1.4. *The case* $\lambda = 0$ *and* $\mathscr{G} \subseteq \mathscr{F}$. In the following proposition we treat the case where $\mathscr{I}_0 \subseteq \mathscr{G} \subseteq \mathscr{E} \subseteq \mathscr{F}$.

Proposition 21. (1) $\mathrm{Pol}(\mathscr{E})$ *is a maximal clone.*

(2) *If* $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ *is a monoid containing all bijections as well as* $\mathscr{I}_0$, *and if* $\mathscr{G} \subseteq \mathscr{E}$, *then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{E})$.

Proof. (1) We prove that for any unary $h \notin \mathscr{E}$ we have $\langle \mathrm{Pol}(\mathscr{E}) \cup \{h\} \rangle = \mathscr{O}$. By definition $h[X]$ is co-large, so we can fix $A \subseteq X$ large and co-large such that $A \cap h[X] = \emptyset$. Choose any $g \in \mathscr{O}^{(1)}$ which maps A onto X and which is constantly $0 \in X$ on $X \setminus A$. Then $g \in \mathscr{E}$ as it is onto. Moreover, $g \circ h$ is constantly 0. Now let an arbitrary $f \in \mathscr{O}^{(n)}$ be given and define a function $\tilde{f} \in \mathscr{O}^{(n+1)}$ by

$$\tilde{f}(x_1, \ldots, x_n, y) = \begin{cases} f(x_1, \ldots, x_n) & , y = 0 \\ y & , \text{otherwise} \end{cases}$$

Then $\tilde{f} \in \mathrm{Pol}(\mathscr{E})$. Indeed, this follows from the inclusion $\tilde{f}(\alpha_1, \ldots, \alpha_n, \beta)[X] \supseteq \beta[X] \setminus \{0\}$ for arbitrary $\alpha_1, \ldots, \alpha_n, \beta \in \mathscr{O}^{(1)}$. Now $f(x) = \tilde{f}(x, 0) = \tilde{f}(x, g \circ h(x_1))$ for all $x \in X^n$ and so $f \in \langle \mathrm{Pol}(\mathscr{E}) \cup \{h\} \rangle$.
(2) Taking an arbitrary $f \notin \mathrm{Pol}(\mathscr{E})$ we show that $f \notin \mathrm{Pol}(\mathscr{G})$. There exist $\alpha_1, \ldots, \alpha_{n_f}$ almost surjective such that $f(\alpha_1, \ldots, \alpha_{n_f})$ is not almost surjective. Consider a small set $A \subseteq X$ so that $A \cup \alpha_i[X] = X$ for all $1 \leq i \leq n_f$. Let γ be a surjection from $X \setminus A$ onto X and define for $1 \leq i \leq n_f$ functions

$$\beta_i(x) = \begin{cases} \alpha_i \circ \gamma(x) & , x \in X \setminus A \\ x & , x \in A \end{cases}$$

Clearly, all β_i are surjective and $f(\beta_1, \ldots, \beta_{n_f})[X] = f(\alpha_1, \ldots, \alpha_{n_f})[X] \cup \{f(x, \ldots, x) : x \in A\}$ is co-large. Fix any $\delta \in \mathscr{I}_0$. Obviously $\beta_i \circ \delta \in \mathscr{I}_0 \subseteq \mathscr{G}$ and also $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta)[X]$ is co-large. Thus $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta) \notin \mathscr{E} \supseteq \mathscr{G}$ so that we infer $f \notin \mathrm{Pol}(\mathscr{G})$. □

In a next step we see what happens in the case $\mathscr{I}_0 \subseteq \mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G} \nsubseteq \mathscr{E}$.

Proposition 22. (1) $\mathrm{Pol}(\mathscr{F})$ *is a maximal clone.*

(2) *If* $\mathscr{G} \subseteq \mathscr{F}$ *is a monoid which contains* $\mathscr{I}_0$ *as well as all bijections, and if* $\mathscr{G} \nsubseteq \mathscr{E}$, *then* $\mathrm{Pol}(\mathscr{G}) \subseteq \mathrm{Pol}(\mathscr{F})$.

Proof. (1) can be found in [7] (Proposition 3.1).
For (2), let $f \notin \mathrm{Pol}(\mathscr{F})$ and fix $\alpha_1, \ldots, \alpha_{n_f} \in \mathscr{F}$ satisfying $f(\alpha_1, \ldots, \alpha_{n_f}) \notin \mathscr{F}$. Since $\mathscr{G} \nsubseteq \mathscr{E}$ but $\mathscr{G} \subseteq \mathscr{F}$, $\mathscr{G}$ must contain a constant function, and hence all constant functions as $\mathscr{S} \subseteq \mathscr{G}$. For those of the α_i which are not constant we construct β_i as in the proof of the preceding proposition, and for the constant ones we set $\beta_i = \alpha_i$. Observe that it is impossible that all α_i are constant. Choosing any $\delta \in \mathscr{I}_0$ we obtain that for all $1 \leq i \leq n_f$, $\beta_i \circ \delta$ is either constant or an element of $\mathscr{I}_0$, and hence in either case an element of $\mathscr{G}$. But as in the preceding proof, $f(\beta_1 \circ \delta, \ldots, \beta_{n_f} \circ \delta) \notin \mathscr{F} \supseteq \mathscr{G}$ so that $f \notin \mathrm{Pol}(\mathscr{G})$. □

2.1.5. *The case* $\lambda = 0$ *and* $\mathscr{G} \nsubseteq \mathscr{F}$. To conclude, we consider submonoids $\mathscr{G}$ of $\mathscr{O}^{(1)}$ which contain the bijections as well as $\mathscr{I}_0$, but which are not submonoids of $\mathscr{F}$. It turns out that the polymorphism clones of such monoids are never maximal. We start with a simple fact about such monoids.

Lemma 23. *Let* $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ *be a monoid containing* $\mathscr{S} \cup \mathscr{I}_0$ *such that* $\mathscr{G} \nsubseteq \mathscr{F}$. *Then* $\mathscr{X} = \{\rho \in \mathscr{O}^{(1)} : |\rho[X]| = 2$ *and* ρ *is generous*$\} \subseteq \mathscr{G}$.

Proof. Let $f \in \mathscr{G} \setminus \mathscr{F}$. Since f is not constant there exist $a \neq b$ in the range of f. Let $s : X \setminus f[X] \to X$ be onto and generous and define $g \in \mathscr{O}^{(1)}$ by

$$g(x) = \begin{cases} s(x) & , x \notin f[X] \\ a & , x = a \\ b & , \text{otherwise} \end{cases}$$

Then $g \in \mathscr{I}_0 \subseteq \mathscr{G}$ and so $g \circ f \circ g \in \mathscr{G}$. On the other hand, $g \circ f \circ g \in \mathscr{X}$ which proves the lemma since obviously any function of $\mathscr{X}$ together with the permutations generate all of $\mathscr{X}$. □

Notation 24. We set $\mathscr{L} = \langle \mathscr{X} \cup \mathscr{I}_0 \cup \mathscr{S} \rangle$. Moreover, we write Const for the set of all constant functions.

The following description of $\mathscr{L}$ is readily verified.

Lemma 25. $\mathscr{L} = \mathrm{Const} \cup \mathscr{X} \cup \mathscr{I}_0 \cup \mathscr{S}$. *In words,* $\mathscr{L}$ *consists exactly of the bijections as well as of all generous functions which are either onto or take at most two values.*

Definition 26. A function $f(x_1, \ldots, x_n) \in \mathscr{O}^{(n)}$ is *almost unary* iff there exist a mapping F from X to the power set of X and some $1 \leq k \leq n$ such that $F(x)$ is small for all $x \in X$ and such that for all $(x_1, \ldots, x_n) \in X^n$ we have $f(x_1, \ldots, x_n) \in F(x_k)$. We denote the set of all almost unary functions by $\mathscr{U}$.

It is easy to see that on a base set of regular cardinality, $\mathscr{U}$ is a clone which contains $\mathscr{O}^{(1)}$. See [5] for a list of all clones above $\mathscr{U}$; there are countably many, so in particular $\mathscr{U}$ is not maximal. The reason for us to consider almost unary functions is the following lemma.

Lemma 27. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ be any function which is not almost unary. Then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Therefore we have

Proposition 28. *If $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ is a nontrivial monoid such that $\mathscr{S} \cup \mathscr{I}_0 \subseteq \mathscr{G}$ and such that $\mathscr{G} \nsubseteq \mathscr{F}$, then $\mathrm{Pol}(\mathscr{G}) \subseteq \mathscr{U}$. In particular, $\mathrm{Pol}(\mathscr{G})$ is not maximal.*

In his proof for countable base sets, Heindorf used the following completeness criterion which has been shown by Gavrilov [1] (Lemma 31 on page 51) to hold on countable base sets; it will follow from our proof of Lemma 27 that this criterion holds on all regular cardinals.

Proposition 29. *Let X have regular cardinality. If $\mathscr{G} \subseteq \mathscr{O}^{(1)}$ is a monoid containing $\mathscr{S} \cup \mathscr{I}_0 \cup \mathscr{X}$, and if $\mathscr{H} \subseteq \mathscr{O}$ is a set of functions such that $\langle \mathscr{O}^{(1)} \cup \mathscr{H} \rangle = \mathscr{O}$, then $\langle \mathscr{G} \cup \mathscr{H} \rangle = \mathscr{O}$.*

The criterion can be used alternatively to show that $\mathrm{Pol}(\mathscr{G})$ is not maximal for the remaining monoids $\mathscr{G}$: We have just seen that $\mathscr{X} \subseteq \mathscr{G}$ so we can apply Proposition 29. Suppose towards contradiction that $\mathrm{Pol}(\mathscr{G})$ is maximal. Since $\mathrm{Pol}(\mathscr{G})^{(1)} = \mathscr{G} \subsetneq \mathscr{O}^{(1)}$ we have $\langle \mathscr{O}^{(1)} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathscr{O}$. But then setting $\mathscr{H} = \mathrm{Pol}(\mathscr{G})$ in the proposition yields that $\langle \mathscr{G} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathscr{O}$, which is impossible as $\langle \mathscr{G} \cup \mathrm{Pol}(\mathscr{G}) \rangle = \mathrm{Pol}(\mathscr{G}) \neq \mathscr{O}$, contradiction.

2.2. **The proof of Lemma 27 and Proposition 29.**

Lemma 30. *Let $u \in \mathscr{O}^{(1)}$ be injective and not almost surjective. Then $\langle \{u\} \cup \mathscr{I}_0 \rangle \supseteq \mathscr{O}^{(1)}$. In particular, $\langle \{u\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Let an arbitrary $f \in \mathscr{O}^{(1)}$ be given. Take any $s : X \setminus u[X] \to X$ which is generous and onto. Now define $g \in \mathscr{O}^{(1)}$ by

$$g(x) = \begin{cases} f(u^{-1}(x)) & , x \in u[X] \\ s(x) & , \text{otherwise} \end{cases}$$

Since $g \restriction_{X \setminus u[X]} = s$ we have $g \in \mathscr{I}_0$. Clearly, $f = g \circ u \in \langle \{u\} \cup \mathscr{I}_0 \rangle$. □

Our strategy for proving Lemma 27 is to show that $\mathscr{L}$ together with a not almost unary f generate a function u as in Lemma 30. We start by observing that $\mathscr{L}$ and f generate functions of arbitrary range.

Lemma 31. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. Then there exists a unary $g \in \langle \{f\} \cup \mathscr{L} \rangle$ such that the range of g is large and co-large.*

Proof. We distinguish two cases.
Case 1. For all $1 \leq i \leq n$ and all $c \in X$ it is true that the image of the hyperplane H^i_c under f is co-small, where $H^i_c = \{x \in X^n : x_i = c\}$. Then consider an arbitrary large and co-large $A \subseteq X$. Set $\Gamma = f^{-1}[X \setminus A] \subseteq X^n$ and let $\alpha : X \to \Gamma$ be onto. By the assumption for this case, $f[H^i_c] \setminus A$ is still large for all $1 \leq i \leq n$ and all $c \in X$. Thus the components $\alpha_i = \pi^n_i \circ \alpha$ are generous and onto; hence, $\alpha_i \in \mathscr{I}_0 \subseteq \mathscr{L}$ for all $1 \leq i \leq n$. But now $f(\alpha_1, \ldots, \alpha_n)[X] = f[X^n] \setminus A$ is large and co-large so that it suffices to set $g = f \circ \alpha$.
Case 2. There exist $1 \leq i \leq n$ and $c \in X$ such that the image $f[H^i_c]$ of the hyperplane H^i_c is co-large, say without loss of generality $i = 1$. Since $f \notin \mathscr{U}$, there exists $d \in X$ satisfying that $f[H^1_d]$ is large. Choose $\Gamma \subseteq X^{n-1}$ large and co-large such that $f[\{d\} \times \Gamma]$ is large and such that $f[H^1_c] \cup f[\{d\} \times \Gamma]$ is still co-large. Take moreover $\alpha_2, \ldots, \alpha_n \in \mathscr{I}_0$ so that $(\alpha_2, \ldots, \alpha_n)[X] = X^{n-1}$. Now we define $\alpha_1 \in \mathscr{O}^{(1)}$ by

$$\alpha_1(x) = \begin{cases} d & , (\alpha_2, \ldots, \alpha_n)(x) \in \Gamma \\ c & , \text{otherwise.} \end{cases}$$

Clearly, $\alpha_1 \in \mathscr{X} \subseteq \mathscr{L}$. Now it is enough to set $g = f(\alpha_1, \ldots, \alpha_n)$ and observe that $g[X] = f[\{c\} \times (X^{n-1} \setminus \Gamma)] \cup f[\{d\} \times \Gamma]$ is large and co-large. $\square$

Lemma 32. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. Then for all nonempty $A \subseteq X$ there exists $h \in \langle \{f\} \cup \mathscr{L} \rangle$ with $h[X] = A$.*

Proof. By Lemma 31 there exists $g \in \langle \{f\} \cup \mathscr{L} \rangle$ having a large and co-large range. Now taking any $\delta \in \mathscr{I}_0 \subseteq \mathscr{L}$ with $\delta[g[X]] = A$ and setting $h = \delta \circ g$ proves the assertion. $\square$

Lemma 33. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$, then $\langle \{f\} \cup \mathscr{L} \rangle$ contains all generous functions.*

Proof. Let any generous $g \in \mathscr{O}^{(1)}$ be given and take with the help of the preceding lemma $h \in \langle \{f\} \cup \mathscr{L} \rangle$ with $h[X] = g[X]$. By setting $h' = h \circ \delta$, where $\delta \in \mathscr{I}_0 \subseteq \mathscr{L}$ is arbitrary, we obtain a generous function with the same property. Now it is clear that there exists a bijection $\sigma \in \mathscr{S} \subseteq \mathscr{L}$ such that $g = h' \circ \sigma$. $\square$

Now that we know that we have all generous functions we want to make them injective. We start by reducing the class of functions f under consideration.

Lemma 34. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ is so that for all $1 \leq i \leq n$ and for all $a, b \in X$ the set of all tuples $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in X^{n-1}$ with $f(x_1, \ldots, x_{i-1}, a, x_{i+1}, \ldots, x_n) \neq f(x_1, \ldots, x_{i-1}, b, x_{i+1}, \ldots, x_n)$ is small, then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Since $f \notin \mathscr{U}$ we can for every $1 \leq i \leq n$ choose $c_i \in X$ such that $f[H^i_{c_i}]$ is large. Choose moreover for every $1 \leq i \leq n$ large sets $A_i \subseteq f[H^i_{c_i}]$ such that $\bigcup_{i=1}^n A_i$ is co-large and such that $A_i \cap A_j = \emptyset$ for $i \neq j$. Write each

A_i as a disjoint union of many large sets: $A_i = \bigcup_{x\in X} A_i^x$. Let $\lhd$ be any well-order of X^n of type κ. Define $\Gamma \subseteq X^n$ by $x \in \Gamma$ iff there exists $1 \leq i \leq n$ such that $f(x) \in A_i^{x_i}$ and whenever $y \lhd x$ and $y \in \Gamma$ then $f(x) \neq f(y)$. Observe that the latter condition ensures that $f \restriction_\Gamma$ is injective.

Now observe that for all $1 \leq i \leq n$, all $d \in X$ and all large $B \subseteq A_i$ we have that $f[H_d^i] \cap B$ is large. Indeed, say without loss of generality $i = 1$ and set $D = \{(x_2, \dots, x_n) : f(d, x_2, \dots, x_n) \neq f(c_1, x_2, \dots, x_n)\}$. Then D is small by our assumption. Now $|f[H_d^1] \cap B| \geq |f[\{d\} \times (X^{n-1} \setminus D)] \cap B| = |f[\{c_1\} \times (X^{n-1} \setminus D)] \cap B| = \kappa$. In particular, this observation is true for $B = A_i^d$. This implies that the set $\{x \in \Gamma : x_i = d\}$ is large for all $1 \leq i \leq n$ and all $d \in X$. Moreover, Γ itself is large.

Therefore there exists a bijection $\alpha : X \to \Gamma$. By the preceding observation, the components $\alpha_i = \pi_i^n \circ \alpha$ are onto and generous, so $\alpha_i \in \mathscr{I}_0 \subseteq \mathscr{L}$ for all $1 \leq i \leq n$. Since α is injective, $\alpha[X] = \Gamma$ and $f \restriction_\Gamma$ is injective, we have that $g = f(\alpha_1, \dots, \alpha_n) \in \langle \{f\} \cup \mathscr{L} \rangle$ is injective. Furthermore, $g[X] = f[\Gamma] \subseteq \bigcup_{i=1}^n A_i$ is co-large. Whence $\mathscr{O}^{(1)} \subseteq \langle \{g\} \cup \mathscr{L} \rangle \subseteq \langle \{f\} \cup \mathscr{L} \rangle$ by Lemma 30 and we are done. □

Lemma 35. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$ is so that for all $1 \leq i \leq n$ there exist $c \in X$ and $S \subseteq H_c^i$ such that $f[S]$ is large and such that for all $b \in X$ the set $\{x \in S : f(x) \neq f(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n)\}$ is small, then $\langle \{f\} \cup \mathscr{L} \rangle \supseteq \mathscr{O}^{(1)}$.*

Proof. Fix for every $1 \leq i \leq n$ an element $c_i \in X$ and a set $S_i \subseteq H_{c_i}^i$ such that $f[S_i]$ large and such that for all $b \in X$ the set $\{x \in S_i : f(x) \neq f(x_1, \dots, x_{i-1}, b, x_{i+1}, \dots, x_n)\}$ is small. Set $A_i = f[S_i]$, $1 \leq i \leq n$. By thinning out the S_i we can assume that the A_i are disjoint and that $\bigcup_{i=1}^n A_i$ is co-large. Now one follows the proof of the preceding lemma. □

Lemma 36. *If $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$, then there exists $g \in \langle \{f\} \cup \mathscr{L} \rangle$ having co-large range and with the property that $\{x \in X : |g^{-1}[x]| = 1\}$ is large (that is, the kernel of g has κ one-element classes).*

Proof. There is nothing to prove if f satisfies the condition of Lemma 35, so assume it does not, and let $i = 1$ witness this. Take $c \in X$ such that $f[H_c^1]$ is large and choose $S \subseteq H_c^1$ with the property that $f[S]$ is still large and that $f \restriction_S$ is injective. By assumption, there exists $b \in X$ such that $\{x \in S : f(x) \neq f(b, x_2, \dots, x_n)\}$ is large. Thus, we can find a large $A \subseteq S$ with the property that $f[A]$ and $f[\{(b, x_2, \dots, x_n) : x \in A\}]$ are disjoint and such that the union of these two sets is co-large. Choose now generous $\alpha_2, \dots, \alpha_n \in \mathscr{O}^{(1)}$ such that $(c, \alpha_2, \dots, \alpha_n)[X] = A$. Since $\langle \{f\} \cup \mathscr{L} \rangle$ contains all generous functions by Lemma 33, we have $\alpha_j \in \mathscr{L}$ for $2 \leq j \leq n$. Take a large and co-large $B \subseteq X$ such that $(c, \alpha_2, \dots, \alpha_n) \restriction_B$ is injective. Define

$$\alpha_1(x) = \begin{cases} c & , x \in B \\ b & , \text{otherwise} \end{cases}$$

and set $g = f(\alpha_1, \dots, \alpha_n)$. Then $g \in \langle \{f\} \cup \mathscr{L} \rangle$ as $\alpha_1 \in \mathscr{X} \subseteq \mathscr{L}$. Clearly, $(\alpha_1, \dots, \alpha_n) \restriction_B$ is injective and so is $g \restriction_B$. Since $g[B]$ and $g[X \setminus B]$ are

disjoint we have that $|g^{-1}[x]| = 1$ for all $x \in g[B]$. Moreover, $g[X] \subseteq f[A] \cup f[\{(b, x_2, \ldots, x_n) : x \in A\}]$ is co-large. $\square$

Lemma 37. *Let $f \in \mathscr{O}^{(n)} \setminus \mathscr{U}$. If $h \in \mathscr{O}^{(1)}$ is a function whose kernel has at least one large equivalence class (that is, there exists $x \in X$ with $h^{-1}[x]$ large), then $h \in \langle \{f\} \cup \mathscr{L} \rangle$.*

Proof. There exist a large $B \subseteq X$ and $b \in X$ such that $h[B] = \{b\}$. Let g be provided by the preceding lemma. With the help of permutations of the base set we can assume that $|g^{-1}[x]| = 1$ for all $x \in g[X \setminus B]$. Since the range of g is co-large we can find $\delta : X \setminus g[X] \to X$ onto and generous. Now define $m \in \mathscr{O}^{(1)}$ by

$$m(x) = \begin{cases} \delta(x) & , x \notin g[X] \\ b & , x \in g[B] \\ h(g^{-1}(x)) & , x \in g[X \setminus B]. \end{cases}$$

Obviously $m \in \mathscr{I}_0 \subseteq \mathscr{L}$ and $h = m \circ g \in \langle \{f\} \cup \mathscr{L} \rangle$. $\square$

Having found many functions which $\langle \{f\} \cup \mathscr{L} \rangle$ must contain, we are finally ready to prove Lemma 27.

Proof of Lemma 27. There are $c_1, \ldots, c_n \in X$ such that $f[H^i_{c_i}]$ is large for $1 \leq i \leq n$. Take for all $1 \leq i \leq n$ large $B_i \subseteq H^i_{c_i}$ with the property that $f \restriction_B$ is injective and $f[B]$ is co-large, where $B = \bigcup_{i=1}^n B_i$. Let $\alpha : X \to B$ be any bijection. Since $\alpha_i^{-1}[c_i]$ is large for every component $\alpha_i = \pi^n_i \circ \alpha$, the preceding lemma yields $\alpha_i \in \langle \{f\} \cup \mathscr{L} \rangle$ for $1 \leq i \leq n$. Whence, $g = f(\alpha_1, \ldots, \alpha_n) \in \langle \{f\} \cup \mathscr{L} \rangle$. But $g[X] = f[B]$ is co-large and g is injective by construction; thus Lemma 30 yields $\mathscr{O}^{(1)} \subseteq \langle \{g\} \cup \mathscr{L} \rangle \subseteq \langle \{f\} \cup \mathscr{L} \rangle$. $\square$

Proof of Proposition 29. Since $\langle \mathscr{O}^{(1)} \cup \mathscr{H} \rangle = \mathscr{O}$, there must exist some $f \in \mathscr{H} \setminus \mathscr{U}$. But then, since $\mathscr{G} \supseteq \mathscr{L}$, Lemma 27 implies $\langle \mathscr{G} \cup \mathscr{H} \rangle \supseteq \mathscr{O}^{(1)}$ so that we infer $\langle \mathscr{G} \cup \mathscr{H} \rangle = \mathscr{O}$. $\square$

3. The proof of Theorem 4

We now determine on an infinite X all maximal submonoids of $\mathscr{O}^{(1)}$ which contain the permutations, proving Theorem 4. In a first section, we present the part of the proof which works on all infinite sets; then follow one section specifically for the case of a base set of regular cardinality and another section for the singular case. Throughout all parts we will mention explicitly whenever a statement is true only on X of regular or singular cardinality, respectively.

3.1. **The part which works for all infinite sets.**

Proposition 38. *$\mathscr{G}_\lambda$ is a maximal submonoid of $\mathscr{O}^{(1)}$ for $\lambda = 1$ and $\aleph_0 \leq \lambda \leq \kappa$.*

Proof. As already mentioned in Lemma 17, the maximality of the $\mathscr{G}_\lambda$ for $\lambda = 1$ or infinite has been proved in [7] (Lemma 5.2). $\square$

The maximal monoids of Proposition 38 already appeared in the preceding section since they give rise to maximal clones via Pol. We shall now expose maximal monoids above the permutations which do not have this property. Recall that $\mathscr{M}_\lambda$ consists of all functions which are either λ-surjective or not λ-injective.

Proposition 39. *Let $\lambda = 1$ or $\aleph_0 \leq \lambda \leq \kappa$. Then $\mathscr{M}_\lambda$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. We show first that $\mathscr{M}_\lambda$ is closed under composition. Let therefore $f, g \in \mathscr{M}_\lambda$, that is, those functions are either λ-surjective or not λ-injective; we claim that $f \circ g$ has either of these properties. It is clear that if g is not λ-injective, then $f \circ g$ has the same property. So let g be λ-surjective. It is easy to see that if f is λ-surjective, then so is $f \circ g$. So assume finally that f is not λ-injective. We claim that $f \circ g$ is not λ-injective either. For $\lambda = 1$ this is just the statement that if f is not injective, and g is surjective, then $f \circ g$ is not injective, which is obvious. Now consider the infinite case. There exist disjoint $A, B \subseteq X$ of size λ such that $f[A] = f[B]$. Set $A' = A \cap g[X]$; A' still has size λ as g misses less than λ values. Clearly $B' = \{x \in B : \exists y \in A'(f(x) = f(y))\}$ has size λ as well and so does $B'' = B' \cap g[X]$. But now for the sets $C = g^{-1}[A']$ and $D = g^{-1}[B'']$ it is true that $|C|, |D| \geq \lambda$, $C \cap D = \emptyset$, and $f \circ g[C] = f \circ g[D]$; hence $f \circ g$ is not λ-injective.
Now we prove that $\mathscr{M}_\lambda$ is maximal in $\mathscr{O}^{(1)}$. Consider for this reason any $m \notin \mathscr{M}_\lambda$, that is, m is λ-injective and misses at least λ values. There exists $A \subseteq X$ so that $|X \setminus A| < \lambda$ and such that the restriction of m to A is injective. Take any injection $i \in \mathscr{O}^{(1)}$ with $i[X] = A$. Then $i \in \mathscr{M}_\lambda$ as i is λ-surjective. Now let $f \in \mathscr{O}^{(1)}$ be arbitrary. Define

$$g(x) = \begin{cases} f((m \circ i)^{-1}(x)) & , x \in m \circ i[X] \\ a & , \text{otherwise} \end{cases}$$

where $a \in X$ is any fixed element of X. Being constant on the complement of the range of m, g it is not λ-injective and whence an element of $\mathscr{M}_\lambda$. Therefore $f = g \circ m \circ i \in \langle \mathscr{M}_\lambda \cup \{m\} \rangle$ so that we infer $\langle \mathscr{M}_\lambda \cup \{m\} \rangle \supseteq \mathscr{O}^{(1)}$. $\square$

Lemma 40. *There are no other maximal monoids above $\mathscr{S} \cup \mathscr{I}_0$ except the $\mathscr{M}_\lambda$ ($\lambda = 1$ or $\aleph_0 \leq \lambda \leq \kappa$).*

Proof. Let $\mathscr{G} \supseteq \mathscr{I}_0 \cup \mathscr{S}$ be a submonoid of $\mathscr{O}^{(1)}$ which is not contained in any of the $\mathscr{M}_\lambda$; we prove that $\mathscr{G} = \mathscr{O}^{(1)}$. To do this, we show that $\mathscr{G}$ contains an injective function $u \in \mathscr{O}^{(1)}$ with co-large range; then the lemma follows from Lemma 30. Fix for every λ a function $m_\lambda \in \mathscr{G} \setminus \mathscr{M}_\lambda$. Since m_κ is κ-injective, there exists a cardinal $\lambda_1 < \kappa$ and a set $A_1 \subseteq X$ of size λ_1 such that the restriction of m_κ to the complement of A_1 is injective. If

λ_1 is infinite, then consider m_{λ_1}. Not being an element of $\mathscr{M}_{\lambda_1}$, m_{λ_1} misses at least λ_1 values. Hence by adjusting it with a suitable permutation we can assume that $m_{\lambda_1}[X] \subseteq X \setminus A_1$. There exists a cardinal $\lambda_2 < \lambda_1$ and a subset A_2 of X of size λ_2 such that the restriction of m_{λ_1} to the complement of A_2 is injective. Hence, writing $\lambda_0 = \kappa$ we obtain that $m_{\lambda_0} \circ m_{\lambda_1} \in \mathscr{G}$ is injective on $X \setminus A_2$ and misses κ values. We can iterate this to arrive after a finite number of steps at a set A_n of finite size λ_n such that the restriction of $m_{\lambda_0} \circ \ldots \circ m_{\lambda_{n-1}} \in \mathscr{G}$ to $X \setminus A_n$ is injective and misses κ values. Since $m_1 \notin \mathscr{M}_1$ is injective and misses at least one value we conclude that the iterate $m_1^{\lambda_n} \in \mathscr{G}$ is injective and misses at least λ_n values. Modulo permutations we may assume that $m_1^{\lambda_n}[X] \subseteq X \setminus A_n$. But now we have that $m_{\lambda_0} \circ \ldots \circ m_{\lambda_{n-1}} \circ m_1^{\lambda_n} \in \mathscr{G}$ is injective and misses κ values, implying that $\mathscr{G} = \mathscr{O}^{(1)}$. □

3.2. **The case of a base set of regular cardinality.** We now finish the proof of Theorem 4 for the case when X has regular cardinality. The proof for this case comprises Propositions 38, 39, 41 and 42.

Proposition 41. *If X is of regular cardinality, then $\mathscr{A}$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. This has been proved in [7] (Proposition 4.1). □

Proposition 42. *Let X have regular cardinality. There exist no other maximal submonoids of $\mathscr{O}^{(1)}$ containing the permutations except those listed in Theorem 4 for the regular case.*

Proof. Assume that $\mathscr{G} \supseteq \mathscr{S}$ is a submonoid of $\mathscr{O}^{(1)}$ not contained in any of the monoids of the theorem; we show that $\mathscr{G} = \mathscr{O}^{(1)}$. Indeed, since $\mathscr{G} \nsubseteq \mathscr{A}$, Proposition 16 tells us that there exists a cardinal $\lambda \leq \kappa$ such that $\mathscr{I}_\lambda$ is contained in $\mathscr{G}$. Choose λ minimal with this property. If λ was greater than 0, then $\mathscr{G} \subseteq \mathscr{G}_\lambda$ for otherwise Lemma 18 would yield a contradiction to the minimality of λ. But this is impossible as we assumed that $\mathscr{G}$ is not contained in any of the $\mathscr{G}_\lambda$, so we conclude that $\lambda = 0$. Now Lemma 40 implies that $\mathscr{G} = \mathscr{O}^{(1)}$. □

3.3. **The case of a base set of singular cardinality.** The only problem with base sets of singular cardinality is that the set $\mathscr{A}$ is not closed under composition; in fact, $\langle \mathscr{A} \rangle = \mathscr{O}$. A slight adjustment of the definition of $\mathscr{A}$ works in this case. We will refer to results from preceding sections; this might look unsafe since there we restricted ourselves to base sets of regular cardinality. However, when proving the particular results cited here we did not use the regularity of the base set. The proof of Theorem 4 for singular cardinals comprises Propositions 38, 39, 46 and 47.

Definition 43. A function $f \in \mathscr{O}^{(1)}$ is said to be *harmless* iff there exists $\lambda < \kappa$ such that the set of all $x \in X$ for which $|f^{-1}[x]| > \lambda$ is small. With this definition, $\mathscr{A}'$ as defined in Theorem 4 is the set of all harmless functions.

Lemma 44. *$\mathscr{A}'$ is a monoid and $\mathscr{A}' \subseteq \mathscr{A}$. Moreover, $\mathscr{A} = \mathscr{A}'$ iff κ is a successor cardinal.*

Proof. It is obvious that $\mathscr{A}' \subseteq \mathscr{A}$ and that $\mathscr{A} = \mathscr{A}'$ iff κ is a successor cardinal. To prove that $\mathscr{A}'$ is closed under composition, let $f, g \in \mathscr{A}'$; we show $h = f \circ g \in \mathscr{A}'$. There exist $\lambda_f, \lambda_g < \kappa$ witnessing that f and g are harmless. Set λ to be $\max(\lambda_f, \lambda_g)$; we claim that the set of $x \in X$ for which $|h^{-1}[x]| > \lambda$ is small. For if $|h^{-1}[x]| > \lambda$, then either $|g^{-1}[x]| > \lambda$ or there exists $y \in g^{-1}[x]$ such that $|f^{-1}[y]| > \lambda$. Both possibilities occur only for a small number of $x \in X$ and so h is harmless. □

Lemma 45. *Let X have singular cardinality. If $g \notin \mathscr{A}'$, then g together with $\mathscr{S}$ generate a function not in $\mathscr{A}$.*

Proof. Set $\lambda < \kappa$ to be the cofinality of κ. Because g is not harmless, there exist distinct sequences $(x^0_\xi)_{\xi<\lambda}, \ldots, (x^\kappa_\xi)_{\xi<\lambda}$ of distinct elements of X such that $\bigcup_{\xi<\lambda} g^{-1}[x^\zeta_\xi]$ is large for all $\zeta < \kappa$. Indeed, if $(\mu_\xi)_{\xi<\lambda}$ is any cofinal sequence of cardinalities in κ, then the fact that g is not harmless allows us to pick for every $\xi < \lambda$ an element $x^0_\xi \in X$ such that $|g^{-1}[x^0_\xi]| > \mu_\xi$; it is also no problem to choose the elements distinct. This yields the first sequence and since with every sequence we are using up only $\lambda < \kappa$ elements, the definition of harmlessness ensures that we can repeat the process κ times. By throwing away half of the sequences, we may assume that the set of all $y \in X$ which do not appear in any of the sequences is large.
There exists a permutation $\alpha \in \mathscr{S}$ such that $g \circ \alpha(x^{\zeta_1}_{\xi_1}) = g \circ \alpha(x^{\zeta_2}_{\xi_2})$ if and only if $\zeta_1 = \zeta_2$, for all $\zeta_1, \zeta_2 < \kappa$ and all $\xi_1, \xi_2 < \lambda$. For we can map every sequence $(x^\zeta_\xi)_{\xi<\lambda}$ injectively into an equivalence class of the kernel of g of size greater than λ; since there are many such classes every sequence can be assigned an own class, and we choose the classes so that a large number of classes are not hit at all. This partial injective mapping we can then extend to the permutation α as it is defined on a co-large set and has co-large range. Set $y^\zeta = g \circ \alpha(x^\zeta_0)$ for all $\zeta < \kappa$. Then the y^ζ are pairwise distinct and for all $\zeta < \kappa$ we have that $(g \circ \alpha \circ g)^{-1}[y^\zeta] \supseteq \bigcup_{\xi<\lambda} g^{-1}[x^\zeta_\xi]$ is large. Hence, $g \circ \alpha \circ g \notin \mathscr{A}$. □

Proposition 46. *Let X have singular cardinality. Then $\mathscr{A}'$ is a maximal submonoid of $\mathscr{O}^{(1)}$.*

Proof. Let $g \in \mathscr{O}^{(1)} \setminus \mathscr{A}'$. We know that g together with $\mathscr{A}'$ generate a function not in $\mathscr{A}$. Then by Lemma 15, we obtain a function which is generous and has large range, call it h. Now take any $f \in \mathscr{O}^{(1)}$ such that $f \circ h[X] = X$ which is injective on $h[X]$ and constant on $X \setminus h[X]$. Then $f \in \mathscr{A}'$ and $f \circ h \in \mathscr{I}_0$. Thus, $\mathscr{I}_0 \subseteq \langle \{g\} \cup \mathscr{A}' \rangle$ and since all injections are elements of $\mathscr{A}'$ we can apply Lemma 30 to prove $\langle \{g\} \cup \mathscr{A}' \rangle \supseteq \mathscr{O}^{(1)}$. □

Proposition 47. *Let X have singular cardinality. There exist no other maximal submonoids of $\mathscr{O}^{(1)}$ containing the permutations except those listed in Theorem 4 for the singular case.*

Proof. If $\mathscr{G} \supseteq \mathscr{S}$ is a submonoid of $\mathscr{O}^{(1)}$ which is not contained in $\mathscr{A}'$, then it is not contained in $\mathscr{A}$ by Lemma 45. From this point, one can follow the proof of Proposition 42. $\square$

References

[1] G. P. Gavrilov, *On functional completeness in countable-valued logic (Russian)*, Problemy Kibernetiki **15** (1965), 5–64.

[2] M. Goldstern and S. Shelah, *Clones on regular cardinals*, Fundam. Math. **173** (2002), no. 1, 1–20.

[3] ———, *Clones from creatures*, Transactions of the American Mathematical Society, to appear.

[4] L. Heindorf, *The maximal clones on countable sets that include all permutations*, Algebra univers. **48** (2002), 209–222.

[5] M. Pinsker, *Clones containing all almost unary functions*, Algebra univers. **51** (2004), 235–255.

[6] I. G. Rosenberg, *Über die funktionale Vollständigkeit in den mehrwertigen Logiken*, Rozpravy Československé Akad. věd, Ser. Math. Nat. Sci. **80** (1970), 3–93.

[7] ———, *Some maximal closed classes of operations on infinite sets*, Math. Annalen **212** (1974), 157–164.

[8] ———, *The set of maximal closed classes of operations on an infinite set A has cardinality* $2^{2^{|A|}}$, Arch. Math. (Basel) **27** (1976), 561–568.

Algebra, TU Wien, Wiedner Hauptstrasse 8-10/104, A-1040 Wien, Austria
E-mail address: marula@gmx.at
URL: http://www.dmg.tuwien.ac.at